ON PARTIALLY BALANCED CHANGE-OVER DESIGNS

G. M. SAHA
Institute of Agricultural Research Statistics, New Delhi
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INTRODUCTION

Designs in which each experimental unit receives a cyclical sequence of several treatments in successive periods, and which estimate direct and residual effects of treatments with varying degrees of precision are known as partially balanced change-over (PBCO) designs. A comprehensive account with a detailed list of balanced change-over (BCO) and PBCO designs for useful ranges of treatments and periods is available in Patterson and Lucas (1962). These designs are 'incomplete blocktype' in the sense that the number of units which are grouped into squares (blocks) is less than v, the number of treatments. Designs of our concern in this paper are, however, 'complete block'-type.

Recently, Davis and Hall (1969) introduced cyclic change-over (CCO) designs as a simple extension of cyclic incomplete block designs [*cf.* John (1966)]. There are 'complete block'-type PBCO designs for v treatments with upto $v/2$ different variances of estimated differences of effects. A table of selected (through the use of computers) CCO designs having maximum efficiencies is appended.

In this paper we have studied two new classes of PBCO designs called 'λ-β-constant' and 'λ-η-constant' designs. Definitions, constructions and analyses of these designs are given in the following sections. These designs as also the derivable extra-period designs (which are obtained from the original designs just by repeating the last-period-treatments in an extra period following the last period) are examined of their efficiencies in terms of the efficiency factors E_d, E_r and E_p of direct, residual and permanent effects of treatments respectively. Certain comparable designs are found to be more efficient than the available PBCO designs. It is also to be noted that the extra-period designs of the new classes estimate residual effects with much higher precision than their ordinary (without extra-period) designs.

2. PBCO Designs

Let, in a change-over design, λ_{ij}, β_j^i and γ_j^i denote respectively:

(*i*) the number of times the treatment pair (i, j) occurs in sequences;

(*ii*) the number of times the treatment pair (i, j) occurs in sequences with j in the last period;

(*iii*) the number of times the treatment j is immediately preceded in sequences by treatment i;

$$i \neq j = 0, 1, 2, \ldots, v-1,$$

where $$(0, 1, 2, \ldots, v-1)$$

symbolise the v treatment in consideration. The normal equations under the usual fixed effects additive model of change-over designs in general involve the parameters λ, β and γ assuming, of course, that a treatment occurs at most once in a sequence and that every treatment occurs equally often in each period. In fact, when λ_{ij}'s, β_j^i's and γ_j^i's are constants for all i and, j and say, equal to λ, β and γ respectively, the designs become balanced. Waiving properly some of the constancy restrictions on the parameters, it is possible to obtain partially balanced designs which require fewer units and at the same time admit of simpler statistical analysis.

Let a design for which $\lambda_{ij}=\lambda$, $\beta_j^i=\beta$ but $\gamma_j^i \neq \gamma$ for all i and j, $i \neq j = 0, 1, \ldots, v-1$; be called a λ-β-constant design, or, in short, $\lambda\beta$ design; and similarly, let a design for which $\lambda_{ij}=\lambda$, $\eta_{ij}=\eta$ where $\eta_{ij}=\beta_j^i+\beta^j_i$ and $\beta^i_j \neq \beta$, $\gamma^i_j \neq \gamma$, for all i and j, $i \neq j = 0, 1, 2, \ldots v-1$ be called a λ-η-constant design or $\lambda\eta$ design.

Two series of $\lambda\beta$ designs and one series of $\lambda\eta$ designs can be constructed through the method of finite differences. We shall present in what follows the required leading sequences of these designs. It may be mentioned at this point that throughout this paper proofs of the theorems and derivations of expressions will be avoided. Interested readers may, however, refer to Saha (1970) for details.

Let $a_1, a_2, \ldots, a_k$ be k distinct elements of mod v the elements of which are $(0, 1, 2, \ldots, v-1)$. Then, a leading sequence

$$(a_1, a_2, \ldots, a_k)$$

will stand for v sequences which can be generated as follows (additions being done mod v) :

		Sequences			
		1	2	...	v
Periods	1	a_1	a_1+1	...	a_1+v-1
	2	a_2	a_2+1	...	a_2+v-1
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	k	a_k	a_k+1	...	a_k+v-1

Also we shall denote a Galois Field of v elements by $GF(v)$ and a primitive root of it by x.

With these definitions and notations, we have, for any v, a prime of the form $4n+1$ $(n>0)$, the following :

Theorem 1. The two leading sequences $(0,\ 2s,\ (2+4)\,s, \ldots (2+4+6+\ldots \overline{v-1})\,s)$ and $(0,\ 2\,a\,s,\ (2+4)\ a\,s, \ldots (2+4+6+\ldots +\overline{v-1})\ as)$ reduced mod v, where s is any non-zero element of $GF(v)$, the v elements of which are represented by $(0, 1, 2, \ldots, v-1)$, and a is any odd power of x, give us a $\lambda\beta$ PBCO design for v treatments in $(v+1)/2$ periods and $2v$ sequences.

When v is a prime power of the form $4n+1$, such designs can be constructed as follows.

Theorem 2. The pair of leading sequences

$(x^0,\ x^2,\ x^4, \ldots, x^{v-3}, 0)$ and $(x^1,\ x^3,\ x^5, \ldots, x^{v-2}, 0)$ reduced mod v, where v is any prime or power of a prime and is of the form $4n+1$ $(n>0)$, generate a $\lambda\beta$ PBCO design for v treatments in $(v+1)/2$ periods and $2v$ sequences.

A series of $\lambda\eta$ designs can be constructed from the following:

Theorem 3. For any prime or prime power v of the form $4n+3$, the leading sequence

$$(x^0, x^2, x^4, \ldots, x^{v-3}, 0) \text{ or } (x^1, x^3, x^5, \ldots, x^{v-2}, 0)$$

reduced mod v gives a $\lambda\eta$ PBCO design for v treatments in v sequences and $(v+1)/2$ periods.

(*Note* : The entries except 0 in the leading sequences of theorems 3 and 4 need not be kept always in the positions shown. They can, rather be randomized over periods before actual experimentation.)

An Example

The method of construction of PBCO designs described in this section is illustrated via an example. Let $v=11$. Here, $x=2$. Theorm 3, then, yields the leading sequence (1, 4, 5, 9, 3, 0) which gives us a $\lambda\eta$ design for 11 treatments in 6 periods and 11 sequences. The sequences are as follows :

Periods	*Sequences*										
1	1	2	3	4	5	6	7	8	9	10	0
2	4	5	6	7	8	9	10	0	1	2	3
3	5	6	7	8	9	10	0	1	2	3	4
4	9	10	0	1	2	3	4	5	6	7	8
5	3	4	5	6	7	8	9	10	0	1	2
6	0	1	2	3	4	5	6	7	8	9	10

3. Analysis of PBCO Designs

The model assumed will be the usual fixed effects one which consists of (*i*) general mean effect, (*ii*) direct and (*iii*) residual effects of treatments, (*iv*) period effect, (*v*) sequence (unit) effect plus a random error component with zero expectations and σ^2 variances. Expressions for least square estimates of direct and residual effects of treatments after adjusting for other effects in $\lambda\beta$ and $\lambda\eta$ designs (which have cyclical solutions only) will be given in this section. Circulant matrices occur in these expressions. Analysis of PBCO designs introduced in this paper requires inversion of circulant matrices. Following Kempthorne (1953) inverses of these matrices of any order can be obtained in general terms. A point to note is that our circulant matrices of coefficients are all non-singular and hence no adjustment, as is usually done, is necessary.

Let t and r denote the $(v\times 1)$ vectors of direct and residual effects of v treatments. Let also P and Q denote the corresponding vectors of "adjusted totals" given by

$$\left.\begin{aligned} P_i &= T_i-(1/K)\sum_{q} S_{q}^{(i)} \\ Q_i &= R_i-(1/K)\sum_{q'} S_{q'}^{(i)}+(P_1'/v-G/vK) \end{aligned}\right\}\ i=0, 1. . ., v-1\ ;$$

where,

(i) T_i and R_i denote respectively the totals of observations which contain (according to the model assumed) direct and residual effects of treatment i ;

(ii) $\sum_q S_q^{(i)}$ denotes the sum of the totals of observations of only those sequence which contain treatment i in any period ; and $\sum_{q'} S_{q'}^{(i)}$, the same for sequences which contain treatment i in any but the final period ;

(iii) P_1' is the total of all observations in the first period and G is the grand total of all the observations ;

(iv) K is the number of periods in each sequence.

Then solving the normal equations under the usual non-contrast linear restrictions on direct, residual, period and sequence effects, we have the solutions for t and r for $\lambda\beta$ designs as

$$t=S^{-1}(\psi P+BQ), \text{ and}$$
$$r=S^{-1}(\xi Q+B'P) ;$$

where,

(i) B is a circulant matrix given by (only the first row will be written in square brackets) ;

$$(B=[\rho, -\gamma^1_0, -\gamma^2_0, \ldots, -\gamma_0^{v-1}],$$

where $\rho=n(K-1)(v-K)/[K(v-1)]$, n being the number of leading sequences of the design, and γ_j^{i}'s as defined earlier;

(ii) $S=(\psi\xi I_v-B'B)$ and S^{-1}, its inverse, where B' is the transpose of B, I_v is the $v\times v$ unit matrix, and $\psi=nv(K-1)/(v-1)$ and $\xi=n(K-1)(vK-v-1)/[K(v-1)]$.

Similar solutions for $\lambda\eta$ designs come out as follows :

$$t=KS^{-1}(\omega P+AQ) \text{ and}$$
$$r=K.S^{-1}(\mu Q+A'P) ;$$

where

(i) $S=(\mu\,\omega\, I_v-A'A)$;

(ii) $A=[\xi, -(\beta_1^0+K\gamma_0^1), -(\beta_2^0+K\gamma_0^2), \ldots, -(\beta^0_{v-1}+K\gamma_0^{v-1})]$

where

$\xi=n(K-1)(v-K-1)/(v-1)$, and γ_j^{i} and β_j^{i}'s are as explained earlier ;

(*iii*) $\omega = K\xi$ and $\mu = K\psi$, ξ and ψ being the quantities as given above.

Analysis of variance tables of $\lambda\beta$ and $\lambda\eta$ designs can be built up on exactly the same lines as of Patterson and Lucas (1962) and hence are omitted here. To partition the treatments S.S. having $2(v-1)$ *d.f.* into two components due to direct and residual effects of treatments each having $(v-1)$ *d.f.*, we shall be requiring the quantities t_i^*'s, and r_i^*'s $(i=0, 1, \ldots, (v-1))$ which denote respectively the estimates of direct effects ignoring residual effects and of residual effects ignoring direct effects of treatments. It is easy to see that for $\lambda\beta$ and $\lambda\eta$ designs, we have,

$$t^*_i = \frac{(v-1)\,P_i}{nv\,(K-1)} \text{ and } r_i^* = \frac{K(v-1)Q_i}{n(K-1)\,(vK-v-1)}\,;$$
$$i=0, 1, \ldots, (v-1).$$

Following Patterson and Lucas (1962), the average efficiency factors E_d, E_r, E_p and E_t of direct, residual, permanent and direct ignoring residual effects have also been worked out for the new classes of designs. We give below the expressions for these quantities.

Let, for the circulant matrices S and B for $\lambda\beta$ designs,

$$S^{-1} = [s_0, s_1, \ldots, s_{v-1}],\ B = [b_0, b_1, \ldots, b_{v-1}]$$

and

$$S^{-1}B = [(sb)_0, (sb)_1, \ldots, (sb)_{v-1}],$$

so that

$$(sb)_0, = \sum_{j=0}^{v-1} s_j b_{v-j}' \ (v-j) \text{ being reduced mod } v.$$

Let also,

$$s. = \sum_{i=0}^{v-1} s_i,\ b. = \sum_{i=0}^{v-1} b_i$$

and

$$(sb). = \sum_{i=0}^{v-1} (sb)_i = s.\, x b.$$

We have, then, for $\lambda\beta$ designs :

$$E_d=(v-1)/[nK\psi\ (vs_0-s.)]\ ;$$

$$E_r=(v-1)/[nK\xi\ (vs_0-s.)]\ ;$$

$$E_p=(v-1)/[nK\ [(\psi+\xi)\ (vs_0-s.)+2\ \{v(sb)_0-(sb).\}]\ ;$$

where ψ and ξ have same meaning as stated earlier.

Using similar notations, we have, for $\lambda\eta$ designs ;

$$E_d=(v-1)/[nK^2\ \omega\ (vs_0-s.)]\ ;$$

$$E_r=(v-1)/[nK^2\mu\ (vs_0-s.)]\ ;$$

$$E_p=(v-1)\ /[nK^2\ [(\omega+\mu)\ (vs_0-s.)+2\{v\ (sa)_0-(sa\}]]\ ;$$

where the matrices S and A and the constants ω and μ are same as those given earlier for $\lambda\eta$ designs. It is easy to see that for both the classes of designs,

$$E_t=[v(K-1)]/[K(v-1)].$$

4. Extra-period PBCO Designs

By repeating the last period treatments of $\lambda\beta$ and $\lambda\eta$ designs in an extra period that follows the last one, extra-period (EP) $\lambda\beta$ and $\lambda\eta$ designs can be constructed. Though these designs do not possess the property of orthogonality between the estimates of direct and residual effects as the balanced change-over designs do, the increase in precision of the estimation of residual effects of these designs is much higher, as compared to that of other extra-period BCO and PBCO designs, than the corresponding increase for the direct effects of treatments.

The least square estimates t and r of the EP $\lambda\beta$ designs can be obtained from the corresponding expressions for ordinary (without extra-period) designs by just replacing ξ, ψ and ρ by their newly defined expressions as given below :

$\xi=[nv(K-1)\ (K+2)]\ /\ [(K+1)\ (v-1)]$, $\psi=[nK(vK-1)]\ /\ [(K+1)(v-1)]$ and $\rho=n(1-K)\ /\ (v-1)$.

These estimates for the EP $\lambda\eta$ designs are, however, to be worked out separately. For these designs, we have, using the notations explained earlier :

$$t=(K+1)S^{-1}\ [\omega P-AQ]\ ;$$

$$r=(K+1)S^{-1}\ [\mu Q-A'P]\ ;$$

where

(*i*) $S=[\mu\ \omega\ I_v-A'A]$,

(*ii*) $A=[\lambda,-\beta_0^{\ 1}-(K+1)\ \gamma_0^{\ 1},-B^2_{\ 0}-(K+1)\ \gamma_0^{\ 2},\ \ldots\ldots -\beta_0^{\ v-1}-(K+1)\ \gamma_0^{\ v-1}]$,

(*iii*) $\mu=(K+1)\ \xi,\ \ \omega=(K+1)\ \psi$ and $\lambda=nK(K-1)/(v-1)\ \xi$ and ψ being the quantities as given above for EP $\lambda\beta$ designs.

Also, it is readily seen that for the extra period designs of both the classes :

$$t_i^*=\frac{(K+1)\ (v-1)P_i}{nv(K-1)\ (K+2)}\ \text{ and } r_i^*=\frac{(K+1)\ (v-1)\ Q_i}{nK(vK-1)}.$$

Expressions for the efficiency factors E_d, E_r, and E_p of EP $\lambda\beta$ and EP $\lambda\eta$ designs are exactly of the same form as those of the corresponding ordinary designs except that K is replaced all through by $(K+1)$ and $+2$ in E_p changes to -2 for extra-period $\lambda\eta$ designs only. Using, therefore, the newly defined constants ξ, ψ, μ and ω in the formulas for E_d, E_r and E_p as given earlier, the average efficiency factors of direct, residual and permanant effects of the extra-period designs of the new classes can be worked out immediately.

Expression for E_t of both the classes of designs is however, simple and is given by :

$$E_t=\frac{v(K-1)\ (K+2)}{(v-1)\ (K+1)^2}.$$

5. List of Designs and Efficiency Factors

We present in this section the leading sequences of some useful designs belonging to the new classes and the corresponding efficiency factors. We have restricted ourselves to $K\leqslant 10$ and $b=nv<100$.

Table 1. Leading sequences for some useful λβ and λη designs.

Captions of different columns of this table are as follows : Col. (1)—Serial No. of designs. Col. (2)—No. of treatment (v). Col. (3)—No. of periods (K). Col. (4)—Total No. of sequences. Col. (5)—The leading sequence (s) Col. (6)—Type of design. (Here, I stands for λβ and II for λη designs ; also, 0 is used for ordinary and E for extra-period designs).

OCLUMNS

1	*2*	*3*	*4*	*5*	*6*
1*	5	3	10	(0,2,1) (0,4,2)	I—0
2	5	4	10	(0,2,1,1) (0,4,2,2)	I—E
3*	7	4	7	(0,1,3,6)	II−0
4	7	5	7	(0,1,3,6,6)	II—E
5*	11	6	11	(0,3,4,8,2,10)	II−0
6	11	7	11	(0,3,4,8,2,10,10)	II—E
7	13	5	39	(0,1,2,4 6) (0,3,5,6,12) (0,3,4,8,11)	II—0
8	13	6	39	(0.1,2,4,6,6,) (0,3,5,6,12,12) (0 3,4,8,11,11)	II−E
9	13	7	26	(0,3,2,11,8,9,12) (0,6,4,9,3,5,11)	I−0
10	13	8	26	(0,3,2,11,8,9,12,12) (0,6,4,9,3,5.11,11)	I—E
11	17	5	68	(0,1,2,7,10) (0,2,3,4,14) (0,4,6,8,11) (0,5,8,12,16,16)	II−0
12	17	6	68	(0,1,2,7,10,10) (0,2,3,4,14,14) (0,4,6,8,11,11) (0,5,8,12,16,16)	II—E
13	17	9	34	(0 2,6,12,3,13,8,5,4) (0,6,1,2,9,5,7,15,12)	I—0
14	17	10	34	(0,2,6,12,3,13,8,5,4,4) (0,6,1,2,9,5,7,15,12,12)	I−E
15	19	4	57	(0,1,4,5) (0,7,9,16) (0,2,8,13)	II−0
16	19	5	57	(0,1,4,5,5) (0,7,9,16,16) (0,2,8,13,13)	II—E
17*	19	10	19	(0,3,15,6,8,16,10,5,4,18)	II—0

*Alternative solutions for leading sequences are :
Design No. 1—(0,2,1) (0,1,3) ;
,, ,, 3—(0,3,2,4) ;
,, ,, 5—(0,6,8,5,4,9) ;
,, ,, 17—(0,6,11,12,16,13,1,10,8,17).

TABLE 2

Efficiency Factors of Designs of Table 1.

Design No. *(See Table 1)*	*Efficiency factors (in percentages)* E_d	E_r	E_p	E_t
1	53	32	14	83
2	72	61	33	78
3	35	25	11	88
4	74	64	34	84
5	75	61	28	92
6	83	74	39	90
7	62	48	23	87
8	68	60	34	84
9	88	74	35	93
10	89	80	42	91
11	65	51	24	85
12	70	62	34	83
13	89	78	37	94
14	90	82	43	94
15	58	42	19	79
16	66	58	31	76
17	87	77	37	95

6. Summary

Two new classes of partially balanced change-over (PBCO) designs with the number of periods less than the number of treatments have been introduced. Methods of construction and analysis of these designs are presented. Extra-period PBCO designs derivable from these new designs are also studied. A table of designs having periods less than or equal to ten and requiring a total number of sequences less than 100 is presented. Different efficiency factors of these designs are also tabulated. It is observed that extra-period designs are much more efficient than the ordinary (without extra period) designs in estimating residual effects. Some of the new designs tabulated are found to possess higher efficiency factors than the existing comparable PBCO designs.

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